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HYPERSINGULAR AND FINITE PART INTEGRALS IN THE BOUNDARY ELEMENT METHOD

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Abstract—A new definition of the Hadamard finite part (HFP) of hypersingular integrals is proposed in this paper. This definition does not involve a limiting process. It is completely general and is valid for one as well as higher dimensional integrals, on closed as well as on open surfaces. It reduces, respectively, to the Cauchy principal value (CPV) and Riemann integral, respectively, for the special cases of strongly singular and weakly singular integrands. Of course, suitable symmetric exclusion zones must be chosen to realize CPV integrals. Starting with this new definition of the HFP of certain hypersingular boundary integral equations (HBIE) that arise in potential theory and in wave scattering, a regularization method is carried out in order to express the hypersingular integrals in terms of ones that are, at most, weakly singular. The regularized versions are completely consistent with those available in the recent literature where a different definition of the HFP was employed.

1. INTRODUCTION

Hypersingular and finite part integrals, and their interpretations and roles in the boundary element method (BEM), have been an important area of research in recent years. This subject has important applications in many areas, the most popular being BEM modeling of scattering of waves by thin scatterers and fracture mechanics (static and dynamic, 2D and 3D).

In general, analytical treatment is required before a hypersingular integral can be evaluated numerically. Several strategies have been proposed by different researchers for this purpose. Some of these are as follows.

- *Integration by parts* (e.g. Sladek and Sladek, 1984; Polch *et al.*, 1987; Nishimura and Kobayashi, 1989). Here, derivatives are transformed from the hypersingular kernels to the functions multiplying them, thereby leaving the kernels at most strongly singular. This approach modifies the primary variables of the boundary integral equations (BIE). These same final formulae have been obtained by other approaches by Zhang and Achenbach (1989).

- *Use of special solutions* (e.g. Rudolphi, 1990, 1991). This approach can be viewed as an extension of the well-known use of rigid body motion solutions for the evaluation of strongly singular integrals in the BEM. This method is very elegant but lacks general applicability (e.g. it fails when one has to deal with open surfaces). This approach has been extended to problems with cracks by Lutz *et al.* (1992).

- *Conversion* (e.g. Krishnasamy *et al.*, 1990; Guiggiani *et al.*, 1992; Bonnet, 1989; Bonnet and Bui, 1993). This is a very general and exact analytical approach that converts hypersingular integrals into ones that can be easily evaluated numerically. The approach [by Krishnasamy *et al.* (1990) and Guiggiani *et al.* (1992)] starts with subtraction and addition of relevant terms to the hypersingular BIE. These terms are

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obtained from Taylor series expansions of the primary variables. This more convenient form of the hypersingular BIE is now converted to integrals that are, at most, weakly singular, by employing Stokes' theorem (Krishnasamy *et al.*, 1990) or by transformation into a parameter plane of intrinsic coordinates followed by further analytical manipulations (Guiggiani *et al.*, 1992). The latter paper above presents numerical evaluations for hypersingular integrals on curved surfaces, for three-dimensional applications of the BIE.

Bonnet and Bui (1993) present two approaches for carrying out this conversion. The first approach, which they call "second order regularization", is analogous to the work described in the previous paragraph. The second approach involves an integration by parts followed by a first order regularization using a variant of Stokes' theorem. Tangential differentiation operators are employed in this work so that the final regularized equations, (from the second approach), involve tractions and tangential derivatives of either the displacements (for the gradient BIE for elasticity) or the crack opening displacements (for the traction BIE for cracks).

- *Direct approach* (e.g. Gray and Soucie, 1993). This method directly uses the hypersingular BIE without any regularization. A source point is first moved away from the boundary of a body. The resulting regular integrals are evaluated analytically, sometimes with the assistance of symbolic computation. A limit process is then performed to bring the source point back to the boundary of the body. For curved elements, this procedure also gives rise to integrals that can be easily evaluated by a combination of analytical integration and usual numerical quadrature (Gray and Soucie, 1993). This direct approach is also exact and powerful.

There exists an intimate relationship between hypersingular boundary integral equations and finite part integrals in the sense of Hadamard (1923). Krishnasamy *et al.* (1990), for example, prove that (provided that the primary variables satisfy certain smoothness requirements) a BIE integral, that becomes hypersingular in the limit as an internal or external source point approaches the boundary, can be interpreted as a Hadamard finite part integral (HFP).

It is the contention of the authors of the present paper that the conventional definition of the HFP integral, as available in the present literature, is not entirely satisfactory and requires modification. Martin (1991), for example, writes the conventional definition

$$I = \neq\!\!\!\!\!\!\int_0^a \frac{f(t)}{(x-t)^2}\,\mathrm{d}t = \lim_{\epsilon\to 0}\left\{\int_0^{x-\epsilon}\frac{f(t)}{(x-t)^2}\,\mathrm{d}t + \int_{x+\epsilon}^{a}\frac{f(t)}{(x-t)^2}\,\mathrm{d}t - \frac{2f(x)}{\epsilon}\right\}, \quad (1)$$

where $0 < x < a$, provided that $f(x)$ is regular in $0 < x < a$ and is a function of the class $C^{1,\beta}$ at $x = t$ for some $\beta > 0$, i.e. $f(t)$ is once differentiable with respect to t in a neighborhood of x and the derivative is Hölder continuous with exponent β in this neighborhood. The inclusion of the last term in eqn (1) is necessary for I to remain bounded. This term is obtained by using a Taylor approximation for $f(t)$ about x and then integrating; the two integrals on the right-hand side of eqn (1) above contribute a term $2f(x)/\epsilon$ which is then cancelled by the last term. A general approach for determining terms such as $-2f(x)/\epsilon$ in eqn (1), that are necessary to keep HFP integrals bounded, appears to be desirable.

Krishnasamy *et al.* (1990) suggest that their eqn (10), obtained by a limiting process of a HBIE equation through the use of a shrinking exclusion zone, can serve to define the HFP of their integral in eqn (5). They also show in this paper that a converted form of their HBIE is equivalent to their equation with HFP integrals. An alternate and direct definition of a HFP, without the need for a limiting process, is proposed in this paper. This definition is valid in any number of dimensions. It is shown that this new definition of the HFP is completely consistent with eqn (14) in Krishnasamy *et al.* (1990) which only contains, at most, weakly singular integrals.

This paper begins with a new definition and discussion of HFP integrals in one dimension. This is followed by a generalization to integrals on curved surfaces in $\mathbb{R}^3$. Some examples are discussed. Next, a connection is made between the present definition of the HFP and the corresponding HBIE. Regularization of the HFP, through the use of Stoke's theorem, follows for an example in potential theory. Finally, an acoustic scattering example is discussed to show that the present definition of the HFP leads to a regularized form of the HBIE that is exactly the same as eqn (14) in Krishnasamy *et al.* (1990).

2. THE HFP IN ONE DIMENSION

Definition 2.1.

Let $\tau\colon I = [-a, b] \to \mathbb{R}$ be a function which has a singularity at $x = 0$ of the form $\tau(x) = O(x^{-\alpha})$ for some $\alpha \in \mathbb{Z}_+$ and that $\phi\colon [-a, b] \to \mathbb{R}$ is a function of the class $C^{\alpha-1,\beta}$ at $x = 0$ for some $\beta > 0$, i.e. ϕ is $(\alpha-1)$ times differentiable in a neighborhood of zero and $\phi^{(\alpha-1)}(x)$ is Hölder continuous with exponent β in this neighborhood. Then the Hadamard finite part of the integral $\int_{-a}^{b}\tau(x)\phi(x)\,\mathrm{d}x$ is defined as the number

$$\oint_{-a}^{b} \tau(x)\phi(x)\,\mathrm{d}x = \langle g_{I_1}, \phi\rangle + \sum_{i=0}^{\alpha-1} \frac{\phi^{(i)}(0)}{i!} f_i(\epsilon_1, \epsilon_2), \tag{2}$$

for any $\epsilon_1, \epsilon_2 > 0$, where

$$\langle g_{I_1}, \phi\rangle = \int_{I \setminus I_1} \tau(x)\phi(x)\,\mathrm{d}x + \int_{I_1} \tau(x)\left[\phi(x) - \sum_{i=0}^{\alpha-1} \frac{\phi^{(i)}(0)}{i!} x^i\right] \mathrm{d}x. \tag{3}$$

Here, I_1 is the interval $I_1 = [-\epsilon_1, \epsilon_2]$ and the functions f_i are defined below.

Note that, in general, $\epsilon_1 \neq \epsilon_2$, and they are not required to be small numbers. Also, since the second integrand above is only weakly singular at $x = 0$ [it is of $O(|x|^{\beta-1})$ as $x \to 0$], $\langle g_{I_1}, \phi\rangle$ is well defined for each $\epsilon_1, \epsilon_2 > 0$.

It is now necessary to define the function $f_i(\epsilon_1, \epsilon_2)$ in eqn (2) in such a way that the definition (2) is independent of the interval I_1. To this end, consider another interval $I_2 = [-\epsilon_1', \epsilon_2']$ which is included in I_1, i.e. $0 < \epsilon_1' < \epsilon_1$ and $0 < \epsilon_2' < \epsilon_2$. One has

$$\begin{aligned}\langle g_{I_1}, \phi\rangle - \langle g_{I_2}, \phi\rangle &= -\sum_{i=0}^{\alpha-1} \frac{\phi^{(i)}(0)}{i!} \int_{I_1 \setminus I_2} \tau(x) x^i\,\mathrm{d}x \\ &= -\sum_{i=0}^{\alpha-1} \frac{\phi^{(i)}(0)}{i!} [f_i(\epsilon_1, \epsilon_2) - f_i(\epsilon_1', \epsilon_2')].\end{aligned} \tag{4}$$

The above equation defines $f_i(\epsilon_1, \epsilon_2)$, which is unique up to an additive constant. The constant is chosen to be zero so that the definition (2) reduces to the usual Riemann integral when $\tau(x)$ is regular.

The definition (2) of the HFP has several properties. It is independent of ϵ_1 and $\epsilon_2 > 0$. The HFP reduces to the strongly singular case when $\tau(x) = O(x^{-1})$. Here, the words "strongly singular" are used to mean integrals that have the same order of singularity as a CPV integral, but are not necessarily CPV in the classical sense. For CPV integrals, the correct result is obtained by setting $\epsilon_1 = \epsilon_2$ in (2). Definition (2) is equivalent to the usual Riemann integral when the integrand is regular. Finally, one could define the HFP by taking the limit $I_1 \to 0$, in which case the second (weakly singular) integral on the right-hand side of eqn (3) would vanish. This is not done here. Instead, it is interesting to consider the other extreme $I_1 = I$, when the first term on the right of eqn (3) would vanish. This question will be considered later.

The remaining important issue is the determination of the function $f_i(\epsilon_1, \epsilon_2)$. This can be carried out analytically in simple cases. In general applications of the hypersingular boundary integral equation method, it should be possible, for example, by integration by parts (for 2D problems) or Stokes' theorem (for 3D problems), to determine such functions. Use of Stokes' theorem for 3D problems is presented later in this paper.

Examples

(1) Let $\tau(x) = |x|^{-p}, p \in \mathbb{Z}_+, I = [-a, b]$. From the equation

$$\int_{-\epsilon_1}^{-\epsilon_1'} \tau(x)x^i \, \mathrm{d}x + \int_{\epsilon_2'}^{\epsilon_2} \tau(x)x^i \, \mathrm{d}x = f_i(\epsilon_1, \epsilon_2) - f_i(\epsilon_1', \epsilon_2'), \tag{5}$$

elementary integration yields

$$f_i(\epsilon_1, \epsilon_2) = \begin{cases} \dfrac{1}{i-p+1}[\epsilon_2^{i-p+1} + (-1)^i \epsilon_1^{i-p+1}] & \text{for } i-p+1 \neq 0 \\ \log(\epsilon_2) + (-1)^i \log(\epsilon_1) & \text{for } i-p+1 = 0 \end{cases} \tag{6}$$

so that the desired integral can now be easily obtained from eqns (2) and (3). As mentioned before, no limit process is involved in determining $f_i(\epsilon_1, \epsilon_2)$ above.

For the case $p = 1$, with $\epsilon = \epsilon_1 = \epsilon_2$ and $\phi(x) = 1$, the result, with $f_0(\epsilon, \epsilon) = 2 \log(\epsilon)$, is

$$J = \mathop{\not{\int}}_{-a}^{b} \frac{\mathrm{d}x}{|x|} = \log(ab).$$

The above is strongly singular but not a CPV integral.

For the case $p = 2$, with $\phi(x) = 1$, $I = [-1, 1]$ and $\epsilon = \epsilon_1 = \epsilon_2$, one gets

$$J = \mathop{\not{\int}}_{-1}^{1} \frac{1}{x^2} \mathrm{d}x = \int_{-1}^{-\epsilon} \frac{1}{x^2} \mathrm{d}x + \int_{\epsilon}^{1} \frac{1}{x^2} \mathrm{d}x + f_0(\epsilon, \epsilon), \tag{7}$$

where $f_0(\epsilon, \epsilon) = -2/\epsilon$ [see eqn (6)]. Also, please see eqn (1).

Taking the limit of the right-hand side as $\epsilon \to 0$, one gets $J = -2$. Letting $\epsilon = 1$ (now there is no limit process involved), one also gets $J = -2$.

(2) This example is taken from Kutt (1975).

Consider

$$J = \mathop{\not{\int}}_{0}^{1} \frac{\mathrm{e}^{-x}}{x-c} \mathrm{d}x, \quad \text{where } c = 0.375. \tag{8}$$

Using eqns (2)–(4), one gets

$$\mathop{\not{\int}}_{0}^{1} \frac{\mathrm{e}^{-x}}{x-c} \mathrm{d}x = \int_{0}^{c-\epsilon_1} \frac{\mathrm{e}^{-x}}{x-c} \mathrm{d}x + \int_{c+\epsilon_2}^{1} \frac{\mathrm{e}^{-x}}{x-c} \mathrm{d}x + \int_{c-\epsilon_1}^{c+\epsilon_2} \frac{\mathrm{e}^{-x} - \mathrm{e}^{-c}}{x-c} \mathrm{d}x + \mathrm{e}^{-c} \log\left(\frac{\epsilon_2}{\epsilon_1}\right). \tag{9}$$

Each of these integrals is, at most, weakly singular, and can be computed by Gaussian quadrature for each chosen pair $\epsilon_1, \epsilon_2 > 0$. The exact value, computable analytically in this case, is $J = -0.3037427810772036$.

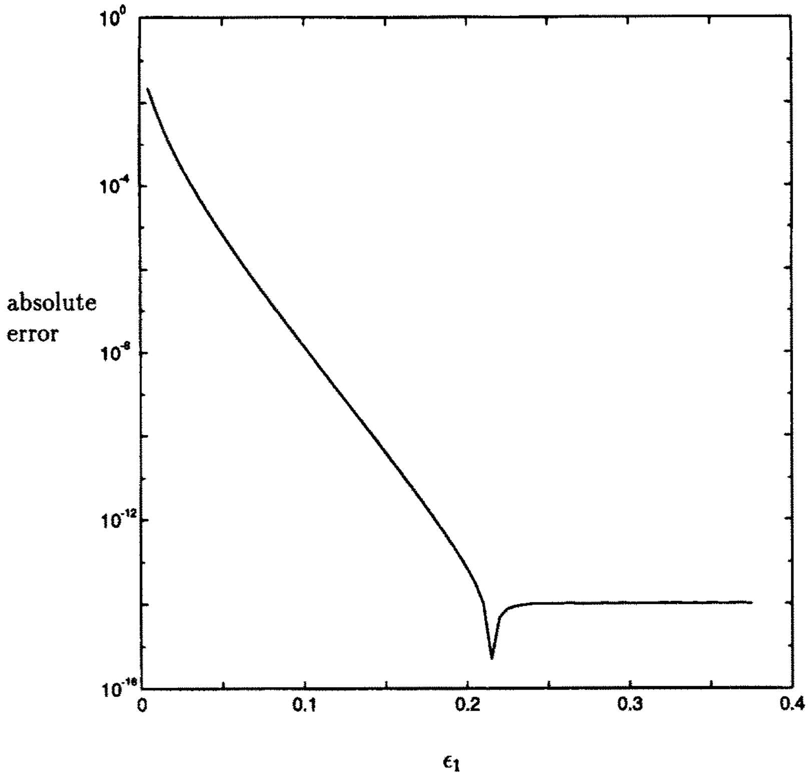

Fig. 1. Error as a function of ϵ_1 for the 1D HFP example.

It is interesting to study the effect of ϵ_1, ϵ_2 on the computed values for J from eqn (9). Figure 1 shows a plot of the error as a function of ϵ_1 (where $\epsilon_2 = 2\epsilon_1$). Eight-point Gaussian quadrature is used here. For a *sufficiently large deleted neighborhood* (i.e. sufficiently large ϵ_1), the numerically obtained result is accurate to at least 12 digits. If $I_1 = I$ (i.e. $\epsilon_1 = 0.375$, $\epsilon_2 = 0.625$), the result is $J = -0.30374278107721$.

It should be pointed out here that the HFP defined by eqns (2) and (3) is independent of I_1. Any dependence of the error on ϵ_1, in Fig. 1, is a consequence of numerical errors arising out of the use of very small values of ϵ_1.

3. THE HFP IN HIGHER DIMENSIONS

Definition 3.1.

Let the space be $\mathbb{R}^3$ and S be a surface in $\mathbb{R}^3$. The points x and ξ_0 lie on $\bar{S}$ (the closure of S) and the points ξ are elsewhere. Let S_1 and S_2 be two neighborhoods of ξ_0 in $\bar{S}$ such that $S_2 \subset S_1$ (Fig. 2).

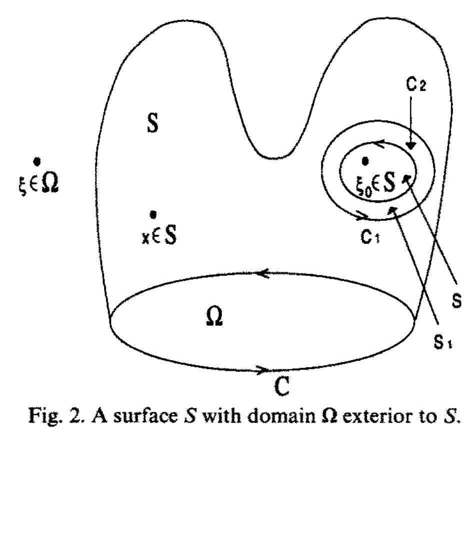

Fig. 2. A surface S with domain Ω exterior to S.

Let the function $\tau: \bar{S} \to \mathbb{R}$ have its only singularity at $x = \xi_0$ of the form

$$\tau(x) = O(|x-\xi_0|^{-r}), \quad \text{where } r \in \mathbb{Z}_+ \text{ and } r \leqslant 3,$$

and let $\phi: \bar{S} \to \mathbb{R}$ be a function which has no singularity in $\bar{S}$ and is of class $C^{1,\beta}$ at ξ_0 for some $\beta > 0$, i.e. ϕ is differentiable in the neighborhood of ξ_0 in $\mathbb{R}^3$ and $|\nabla\phi(x) - \nabla\phi(y)| \leqslant M|x-y|^\beta$ for a fixed constant $M > 0$ and all x, y in this neighborhood.

The Hadamard finite part of $\int_{\bar{S}} \tau(x)\phi(x)\, \mathrm{d}S(x)$ is defined as

$$⨎_{\bar{S}} \tau(x)\phi(x)\, \mathrm{d}S(x) = \langle g_{S_1}, \phi\rangle + \phi(\xi_0)A(S_1) + \phi_{,p}(\xi_0)B_p(S_1), \tag{10}$$

for any neighborhood S_1 of ξ_0 in $\bar{S}$. Here,

$$\langle g_{S_1}, \phi\rangle = \int_{\bar{S}\setminus S_1} \tau(x)\phi(x)\, \mathrm{d}S(x) + \int_{S_1} \tau(x)[\phi(x) - \phi(\xi_0) - \phi_{,p}(\xi_0)(x_p - \xi_{0p})]\, \mathrm{d}S(x). \tag{11}$$

Now consider the expression

$$\begin{aligned}\langle g_{S_1}, \phi\rangle - \langle g_{S_2}, \phi\rangle &= -\int_{S_1\setminus S_2} \tau(x)[\phi(\xi_0) + \phi_{,p}((\xi_0)(x_p - \xi_{0p})]\, \mathrm{d}S(x) \\ &= -\phi(\xi_0)\int_{S_1\setminus S_2} \tau(x)\, \mathrm{d}S(x) - \phi_{,p}(\xi_0)\int_{S_1\setminus S_2} \tau(x)(x_p - \xi_{0p})\, \mathrm{d}S(x) \\ &= -\phi(\xi_0)[A(S_1) - A(S_2)] - \phi_{,p}(\xi_0)[B_p(S_1) - B_p(S_2)].\end{aligned} \tag{12}$$

Thus,

$$\int_{S_1\setminus S_2} \tau(x)\, \mathrm{d}S(x) = A(S_1) - A(S_2) \tag{13}$$

and

$$\int_{S_1\setminus S_2} \tau(x)(x_p - \xi_{0p})\, \mathrm{d}S(x) = B_p(S_1) - B_p(S_2). \tag{14}$$

Equations (13) and (14) define $A(S_1)$ and $B_p(S_1)$, respectively.

As before for the 1D case, the HFP defined by eqn (10) has several interesting properties. It is independent of the choice of the deleted neighborhood S_1 of ξ_0. The terms $A(S_1)$ and $B_p(S_1)$ can be identified, in some sense, with the expressions

$$\int_{S_1} \tau(x)\, \mathrm{d}S(x), \quad \int_{S_1} \tau(x)(x_p - \xi_{0p})\, \mathrm{d}S(x), \tag{15}$$

respectively, even though the above integrals do not exist. If $r = 2$, the HFP reduces to the strongly singular case. For CPV integrals, one must further choose the deleted neighborhood to be a disk centered at ξ_0 in $\bar{S}$. If $\tau(x)$ has no singularity in $\bar{S}$, the HFP reduces to the usual Riemann integral. On the other hand, if $\tau(x) = O(|x-\xi_0|^{-r})$ and $r \in \mathbb{Z}_+$ is greater than three (should such integrals be called supersingular?), the above definition can be extended provided that ϕ is of the class $C^{r-2,\beta}$ at $x = \xi_0$ for some $\beta > 0$. Now, higher order terms [up to the $(r-2)$th derivative] must be included in the Taylor expansion of $\phi(x)$ at $x = \xi_0$. Finally, the HFP in eqn (10) can be evaluated once the functions $A(S_1)$ and $B_p(S_1)$ are determined, since only weakly singular kernels would have to be numerically computed.

These functions $A(S_1)$ and $B_p(S_1)$ can be determined analytically in simple cases, or by the use of Stokes' theorem in general applications of the HBIE.

Example

Consider

$$J = \oint_{S} \frac{e^{-r}}{r^3} dS(x), \tag{16}$$

where $\bar{S}$ is a closed unit disk centered at the origin and r is its generic radius. From eqn (10), one gets

$$J = \int_{S \setminus S_1} \frac{e^{-r}}{r^3} dS(x) + \int_{S_1} \frac{e^{-r} - 1 + r}{r^3} dS(x) + A(S_1) - B(S_1), \tag{17}$$

where S_1 is chosen to be a disk of radius r_1 centered at the origin. Here $\xi_0 = 0$ and $B = B_1 + B_2$. From eqn (13), one has

$$A(S_1) - A(S_2) = \int_{S_1 \setminus S_2} \frac{dS(x)}{r^3} = 2\pi \int_{r_2}^{r_1} \frac{dr}{r^2} = -2\pi \left[\frac{1}{r_1} - \frac{1}{r_2} \right], \tag{18}$$

so that $A(S_1) = -2\pi/r_1$. Similarly, $B(S_1) = 2\pi \log(r_1)$. Finally,

$$\oint_{S} \frac{e^{-r}}{r^3} dS(x) = 2\pi \int_{r_1}^{1} \frac{e^{-r}}{r^2} dr + 2\pi \int_{0}^{r_1} \frac{e^{-r} - 1 + r}{r^2} dr - \frac{2\pi}{r_1} - 2\pi \log(r_1). \tag{19}$$

The exact HFP is $-2\pi(0.571279841\,87439)$. The error in the computed HFP [using eqn (19) with eight-point Gaussian quadrature] is plotted as a function of r_1 in Fig. 3. Again,

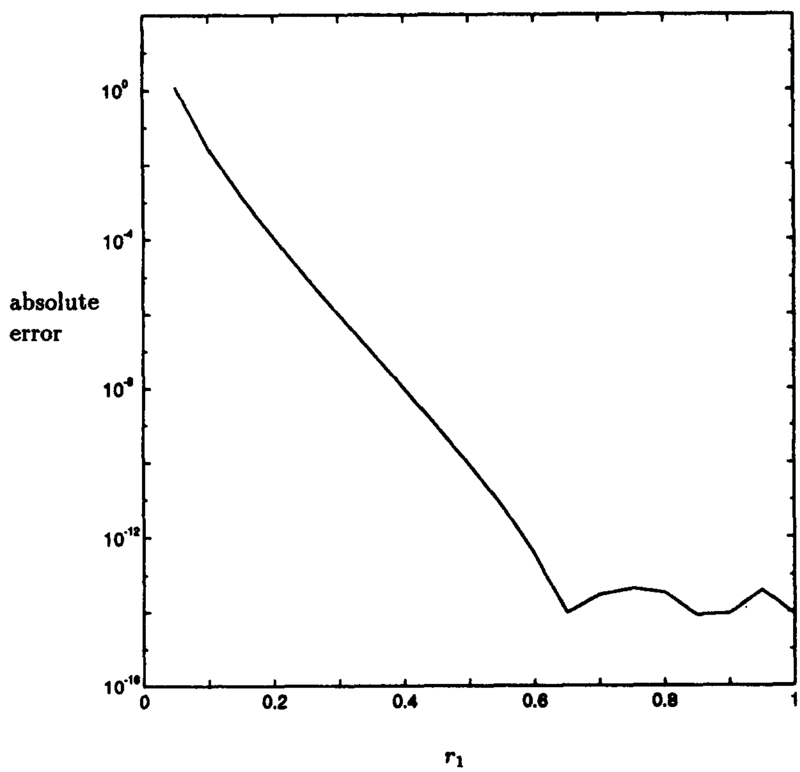

Fig. 3. Error as a function of r_1 for the 2D HFP example.

for a *sufficiently large deleted neighborhood of the singularity*, the numerical value is correct to at least 12 digits. If $S_1 = \bar{S}$ (i.e. $r_1 = 1$), the numerical value is $-2\pi(0.57127984187439)$. Note that eqn (19) fails if one tries to numerically evaluate its value for $r_1 \to 0$. Of course, such is not the intention here.

4. THE HFP AND THE HBIE

Let S be a surface in $\mathbb{R}^3$. Let S^+ and S^- be the two sides of S. Points x and ξ_0 lie on S^+ or S^- and points ξ are elsewhere (see Fig. 2). Consider the expression

$$I(\xi) = \int_{S^+} K(x,\xi)u(x)\,\mathrm{d}S(x), \quad \xi \text{ not in } S. \tag{20}$$

Let $K(x,\xi)$ have its only singularity at $\xi = \xi_0$ of the form

$$K(x,\xi) = O(|x-\xi|^{-3}). \tag{21}$$

Proposition 4.1.
If $u \in C^{1,\beta}$ at $x = \xi_0$, then

$$\lim_{\xi \to \xi_0} I(\xi) = \,=\!\!\!\!\!\!\int_{S^+} K(x,\xi_0)u(x)\,\mathrm{d}S(x). \tag{22}$$

Krishnasamy *et al.* (1990) essentially give a proof of the above proposition for the case of wave scattering. It is important, however, to show that the above proposition holds with the new definition of the HFP that has been proposed in the present work. A proof of this proposition is given in Appendix A.

5. REGULARIZATION USING STOKES' THEOREM

Consider, for illustrative purposes, a function $u(x)$ that satisfies the Laplace equation in the closed domain $\Omega \cup S$, where Ω is an open set in $\mathbb{R}^3$ and S is the closed surface bounding Ω (see Fig. 4). Starting with

$$\nabla^2 u(\xi) = 0, \quad \text{for } \xi \in \Omega, \tag{23}$$

the usual BEM representation for $u(\xi)$ is

$$u(\xi) = \int_S \left[G(x,\xi)\frac{\partial u(x)}{\partial n(x)} - \frac{\partial G(x,\xi)}{\partial n(x)} u(x) \right] \mathrm{d}S(x), \tag{24}$$

where $G(x,\xi) = 1/4\pi R$, $R = |x-\xi|$, $x \in S$ and n is the unit outward normal to S at a point on it.

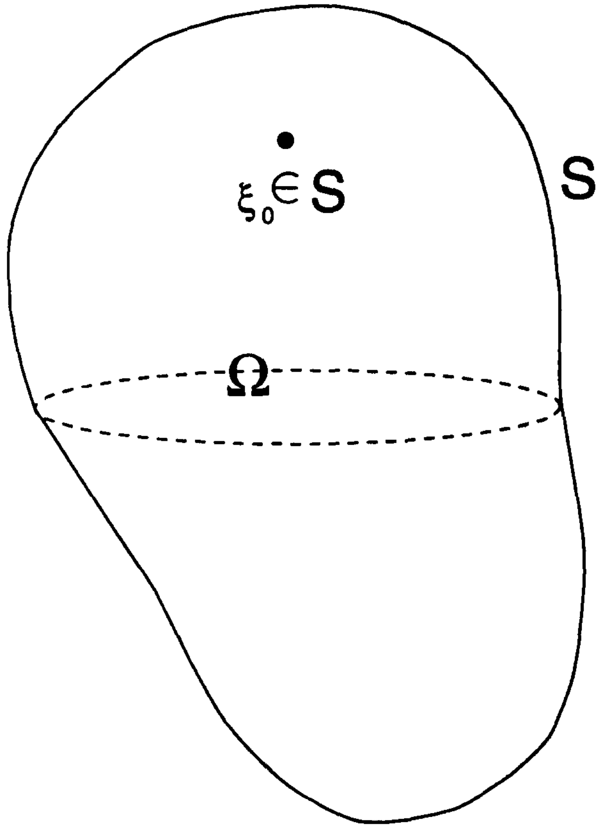

Fig. 4. An open set Ω with closed bounding surface S.

The well-known gradient form of the above equation is

$$\frac{\partial u(\xi)}{\partial \xi_r} = \int_S \left[\frac{\partial G(x,\xi)}{\partial \xi_r} \frac{\partial u(x)}{\partial n(x)} - \frac{\partial^2 G(x,\xi)}{\partial \xi_r \, \partial n(x)} u(x) \right] \mathrm{d}S(x). \tag{25}$$

Proposition 5.1.
Let $\xi \in S$, then

$$\begin{aligned} \frac{\partial u(\xi_0)}{\partial n(\xi_0)} &= n_r(\xi_0) \lim_{\xi \to \xi_0} \frac{\partial u(\xi)}{\partial \xi_r} \\ &= n_r(\xi_0) \,\, \not{\!\!\!\!\int}_S \left[\frac{\partial G(x,\xi_0)}{\partial \xi_r} \frac{\partial u(x)}{\partial n(x)} - \frac{\partial^2 G(x,\xi_0)}{\partial \xi_r \, \partial n(x)} u(x) \right] \mathrm{d}S(x), \end{aligned} \tag{26}$$

provided that $u \in C^{1,\beta}$ at $x = \xi_0$ for some $\beta > 0$ and ξ_0 does not lie at a corner of S.

The proof of Proposition 5.1 can be carried out as follows.

(1) Apply Proposition 4.1 to find the limits, as $\xi \to \xi_0$, for the two integrals on the right-hand side of eqn (25).
(2) Take the inner product of each of these equations with $n_r(\xi_0)$.
(3) Use the inner product of (25) with $n_r(\xi_0)$.

Generally, analytical evaluation of the right-hand side of eqn (26) is not possible. The integral

$$\frac{\partial u(\xi_0)}{\partial \xi_r} = \lim_{\xi \to \xi_0} \frac{\partial u(\xi)}{\partial \xi_r}$$
$$= \oint_S \left[\frac{\partial G(x,\xi_0)}{\partial \xi_r} \frac{\partial u(x)}{\partial n(x)} - \frac{\partial^2 G(x,\xi_0)}{\partial \xi_r \, \partial n(x)} u(x) \right] \mathrm{d}S(x), \tag{27}$$

however, can be expressed in terms of weakly singular integrals through the use of Stokes' theorem. Once this is done, the above integral can be easily evaluated numerically.

The final result is given below (see Fig. 4). Details of the derivation are given in Appendix B.

$$\begin{aligned} \frac{\partial u(\xi_0)}{\partial \xi_r} = & \int_{S \setminus S_1} \left[\frac{\partial G(x,\xi_0)}{\partial \xi_r} \frac{\partial u(x)}{\partial n(x)} - \frac{\partial^2 G(x,\xi_0)}{\partial \xi_r \, \partial n(x)} u(x) \right] \mathrm{d}S(x) \\ & + \int_{S_1} \frac{\partial G(x,\xi_0)}{\partial \xi_r} \left[\frac{\partial u(x)}{\partial n(x)} - \frac{\partial u(\xi_0)}{\partial n(\xi_0)} \right] \mathrm{d}S(x) \\ & - \int_{S_1} \frac{\partial^2 G(x,\xi_0)}{\partial \xi_r \, \partial n(x)} [u(x) - u(\xi_0) - u_{,p}(\xi_0)(x_p - \xi_{0p})] \, \mathrm{d}S(x) \\ & + u_{,p}(\xi_0) \left[\epsilon_{qrp} \oint_{C_1} G(x,\xi_0) \, \mathrm{d}x_q + \epsilon_{qkr} \oint_{C_1} G_{,k}(x,\xi_0)(x_p - \xi_{0p}) \, \mathrm{d}x_q \right. \\ & \left. + \int_{S_1} G_{,r}(x,\xi_0)[n_p(x) - n_p(\xi_0)] \, \mathrm{d}S(x) \right] \\ & + u(\xi_0)\epsilon_{qkr} \oint_{C_1} G_{,k}(x,\xi_0) \, \mathrm{d}x_q + \frac{u_{,r}(\xi_0)}{4\pi} \Omega_{S_1}(\xi_0). \end{aligned} \tag{28}$$

Here, $\Omega_{S_1}(\xi_0)$ is the solid angle subtended by the surface S_1 at ξ_0. Also, ϵ_{qkr} is the alternating symbol and $_{,k} \equiv \partial/\partial x_k$. The geometrical symbols S, S_1 and C_1 are shown in Fig. 4.

The integrals in (28) are, at most, weakly singular, provided that $n(x)$ is of class $C^{0,\beta}$ at ξ_0 for some $\beta > 0$. The exclusion zone S_1 is arbitrary in problems with closed surfaces. Numerical experience with 1D integrals, presented earlier in this paper, suggests the use of a *sufficiently large exclusion zone for accurate numerical evaluation of the HFP integrals*. In problems with open surfaces, S_1 can be set equal to the entire open surface S. This is illustrated in the scattering example in the next section.

6. SCATTERING OF ACOUSTIC WAVES

Krishnasamy *et al.* (1990) present an example of scattering of acoustic waves by a thin rigid scatterer (Fig. 2). In this paper, they consider, in detail, regularization of the HFP integral

$$I_n(\xi_0) = -\oint_{S^+} \frac{\partial^2 G(x,\xi_0)}{\partial n(x)\, \partial n(\xi_0)} u(x) \, \mathrm{d}S(x), \tag{29}$$

where G and n are the same as in the previous section and $u = \phi^+ - \phi^-$ with $\phi = \phi^i + \phi^s$, the sum of the incident and scattered fields. For a rigid scatterer, $\partial\phi^+/\partial n = \partial\phi^-/\partial n = 0$, therefore $\partial u/\partial n = 0$.

Regularization proceeds in the same manner as in the previous section. One gets

$$\oint_{S^+} \frac{\partial^2 G(x,\xi_0)}{\partial \xi_r \, \partial n(x)} u(x) \, dS(x) = \int_{S^+ \setminus S_1} \frac{\partial^2 G(x,\xi_0)}{\partial \xi_r \, \partial n(x)} u(x) \, dS(x)$$

$$+ \int_{S_1} \frac{\partial^2 G(x,\xi_0)}{\partial \xi_r \, \partial n(x)} [u(x) - u(\xi_0) - u_{,p}(\xi_0)(x_p - \xi_{0p})] \, dS(x)$$

$$+ u(\xi_0) A_r(S_1) + u_{,p}(\xi_0) B_{rp}(S_1), \quad (30)$$

where the functions $A_r(S_1)$ and $\{B_{rp}(S_1) - B_{rp}(S_2)\}$ are given by eqns (B3) and (B4) (see Appendix B), respectively.

Using eqn (B4), one gets

$$-u_{,p}(\xi_0) B_{rp}(S_1) = u_{,p}(\xi_0) \int_{S_1} G_{,r}(x,\xi_0)[n_p(x) - n_p(\xi_0)] \, dS(x)$$

$$+ \int_{S_1} G_{,r}(x,\xi_0) u_{,p}(\xi_0) n_p(\xi_0) \, dS(x) + \frac{u_{,r}(\xi_0)}{4\pi} \Omega_{S_1}(\xi_0)$$

$$+ u_{,p}(\xi_0) \left[\epsilon_{qkr} \oint_{C_1} G_{,k}(x,\xi_0)(x_p - \xi_{0p}) \, dx_q \right.$$

$$\left. + \epsilon_{qrp} \oint_{C_1} G(x,\xi_0) \, dx_q \right]. \quad (31)$$

The first integrand on the right-hand side of the above equation is weakly singular if $n(x)$ is of class $C^{0,\beta}$ at ξ_0 for some $\beta > 0$. The second term vanishes for a rigid scatterer where $\partial u(\xi_0)/\partial n(\xi_0) = u_{,p}(\xi_0) n_p(\xi_0) = 0$. It is important to note here that while some terms in B_{rp} and C_r may be strongly singular, and therefore undefined for general neighborhoods S of ξ_0, the combination of terms that occur in the Laplace equation and scattering examples are, at most, weakly singular, and therefore easily computable.

Finally, taking the inner product of both sides of eqn (30) with $n_r(\xi_0)$, using eqns (31) and (B3), noting that $\partial u(\xi_0)/\partial n(\xi_0) = 0$, and *setting* $S_1 = S^+$, results in an expression for $I_n(\xi_0)$ which is identical to eqn (14) of Krishnasamy *et al.* (1990) except for one term. The term in question is:

Krishnasamy *et al.* (1990): $$-u_{,k}(\xi_0) n_r(\xi_0) \int_S \frac{\partial G(x,\xi_0)}{\partial \xi_r} n_k(x) \, dS(x)$$

here: $$-u_{,k}(\xi_0) n_r(\xi_0) \int_S \frac{\partial G(x,\xi_0)}{\partial \xi_r} [n_k(x) - n_k(\xi_0)] \, dS(x).$$

The latter expression is a regularized version of the former. The two expressions are equivalent in view of the fact that, for a rigid scatterer, $\partial u(\xi_0)/\partial n(\xi_0) = 0$.

7. CONCLUSIONS

The new definition of the HFP of hypersingular integrals, proposed in this paper, is completely unified and general. It avoids the common practice of defining the HFP by somehow ignoring parts of an integral that blow up and retaining the rest. Such definitions often tend to be problem specific and difficult to generalize. It is also shown in the present paper that the proposed new definition of the HFP is consistent with recent results, reported in the literature, on regularized HBIEs that appear in problems of scattering of waves by extremely thin scatterers.

Referring back to the general definition of the HFP in one [eqns (2) and (3)] or in the higher dimensions [eqns (10) and (11)], it is proposed that when the domain of the integral is an open region (curve or surface), the exclusion zone S_1 should be set equal to the entire (curve or) surface S. For closed surfaces, the exclusion zone, based on preliminary numerical results for 1D integrals, should be made sufficiently large.

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APPENDIX A

Proof of Proposition 4.1. (see Fig. 2)

This proof follows directly from the definition (A1) below of an HFP integral, and certain continuity arguments.

$$\oint_{S^+} K(x,\xi_0)u(x)\,\mathrm{d}S(x) = \int_{S^+\setminus S_1} K(x,\xi_0)u(x)\,\mathrm{d}S(x) + \int_{S_1} K(x,\xi_0)[u(x)-u(\xi_0)-u_{,p}(\xi_0)(x_p-\xi_{0p})]\,\mathrm{d}S(x)$$
$$+u(\xi_0)A(S_1)+u_{,p}(\xi_0)B_p(S_1), \quad \text{(A1)}$$

where S_1 is a neighborhood of ξ_0 in S^+ (note that S^+ and S^- are the two sides of S).

$$A(S_1)-A(S_2) = \int_{S_1\setminus S_2} K(x,\xi_0)\,\mathrm{d}S(x) = \lim_{\xi\to\xi_0} \int_{S_1\setminus S_2} K(x,\xi)\,\mathrm{d}S(x).$$

The second equality holds because $K(x,\xi_0)$ has no singularity in $S_1\setminus S_2$. Assuming that the limit $\lim_{\xi\to\xi_0}\int_{S_i} K(x,\xi)\,\mathrm{d}S(x)$ exists for $i = 1, 2$, then

$$A(S_1) = \lim_{\xi\to\xi_0} \int_{S_1} K(x,\xi)\,\mathrm{d}S(x). \quad \text{(A2)}$$

By a similar argument, one has

$$B_p(S_1) = \lim_{\xi\to\xi_0} \int_{S_1} K(x,\xi)(x_p-\xi_p)\,\mathrm{d}S(x), \quad \text{for each } p. \quad \text{(A3)}$$

Next, one considers the first and second terms of eqn (A1). Since the integrands have no singularity in the domain of integration, by continuity, one has

$$\int_{S^+\backslash S_1} K(x,\xi_0)u(x)\,dS(x) = \lim_{\xi\to\xi_0}\int_{S^+\backslash S_1} K(x,\xi)u(x)\,dS(x), \tag{A4}$$

and

$$\int_{S^+\backslash S_1} K(x,\xi_0)[u(x)-u(\xi_0)-u_{,p}(\xi_0)(x_p-\xi_{0p})]\,dS(x) = \lim_{\xi\to\xi_0}\int_{S^+\backslash S_1} K(x,\xi)[u(x)-u(\xi)-u_{,p}(\xi)(x_p-\xi_p)]\,dS(x). \tag{A5}$$

Combining (A2)–(A5), Proposition 4.1. is proved.

APPENDIX B

First, it should be noted here that Stokes' theorem is used in the form

$$\int_S [F_{,r}(x)n_k(x)-F_{,k}(x)n_r(x)]\,dS(x) = \oint_C \epsilon_{qkr}F(x)\,dx_q, \tag{B1}$$

where (see Fig. 2) S is an open surface in $\mathbb{R}^3$ and C is the curve that bounds S. Also, F is a differentiable function defined on an open neighborhood of $S \cup C$ in $\mathbb{R}^3$ and takes values in $\mathbb{R}^3$.

Applying the definition of the HFP [eqns (10) and (11)], one gets

$$\begin{aligned}
&\neq\!\!\!\!\!\!\int_S \left[\frac{\partial G(x,\xi_0)}{\partial \xi_r}\frac{\partial u(x)}{\partial n(x)} - \frac{\partial^2 G(x,\xi_0)}{\partial \xi_r\,\partial n(x)}u(x)\right]dS(x)\\
&= \int_{S\backslash S_1}\left[\frac{\partial G(x,\xi_0)}{\partial \xi_r}\frac{\partial u(x)}{\partial n(x)} - \frac{\partial^2 G(x,\xi_0)}{\partial \xi_r\,\partial n(x)}u(x)\right]dS(x) + \int_{S_1}\frac{\partial G(x,\xi_0)}{\partial \xi_r}\left[\frac{\partial u(x)}{\partial n(x)} - \frac{\partial u(\xi_0)}{\partial n(\xi_0)}\right]dS(x)\\
&\quad - \int_{S_1}\frac{\partial^2 G(x,\xi_0)}{\partial \xi_r\,\partial n(x)}[u(x)-u(\xi_0)-u_{,p}(\xi_0)(x_p-\xi_{0p})]\,dS(x)\\
&\quad - u(\xi_0)A_r(S_1) - u_{,p}(\xi_0)[B_{rp}(S_1)-n_p(\xi_0)C_r(S_1)],
\end{aligned} \tag{B2}$$

where S_1 is a neighborhood of ξ_0 in S (see Fig. 2) and the functions A_r, B_{rp} and C_r are determined below. It should be noted that A_r and B_{rp} above correspond to the functions in eqn (10) while the function C_r arises from the strongly singular kernel [the term $\partial G(x,\xi_0)/\partial\xi_r$) in eqn (27)].

Determination of $A_r(S_1)$ (see Fig. 2)

From eqn (13)

$$\begin{aligned}
A_r(S_1)-A_r(S_2) &= \int_{S_1\backslash S_2}\frac{\partial^2 G(x,\xi_0)}{\partial \xi_r\,\partial n(x)}\,dS(x)\\
&= -\int_{S_1\backslash S_2} G_{,kr}(x,\xi_0)n_k(x)\,dS(x).
\end{aligned}$$

Using the fact that $\nabla^2 G(x,\xi_0) = 0$ for $x \neq \xi_0$,

$$A_r(S_1)-A_r(S_2) = -\int_{S_1\backslash S_2}[G_{,kr}(x,\xi_0)n_k(x)-G_{,kk}(x,\xi_0)n_r(x)]\,dS(x).$$

Denoting the positively oriented boundary of $S_1\backslash S_2$ by $C_1\backslash C_2$ (see Fig. 2) and applying Stokes' theorem [eqn (B1)], one gets

$$A_r(S_1)-A_r(S_2) = -\epsilon_{qkr}\oint_{C_1\backslash C_2} G_{,k}(x,\xi_0)\,dx_q,$$

so that

$$A_r(S_1) = -\epsilon_{qkr}\oint_{C_1} G_{,k}(x,\xi_0)\,\mathrm{d}x_q. \tag{B3}$$

Determination of $B_{rp}(S_1)$

From (14),

$$\begin{aligned} B_{rp}(S_1)-B_{rp}(S_2) &= \int_{S_1\setminus S_2} \frac{\partial^2 G(x,\xi_0)}{\partial \xi_r\,\partial n(x)}(x_p-\xi_{0p})\,\mathrm{d}S(x) \\ &= -\int_{S_1\setminus S_2} G_{,kr}(x,\xi_0)(x_p-\xi_{0p})n_k(x)\,\mathrm{d}S(x) \\ &= -\int_{S_1\setminus S_2} [G_{,k}(x,\xi_0)(x_p-\xi_{0p})]_{,r}n_k(x)\,\mathrm{d}S(x) \\ &\quad +\delta_{rp}\int_{S_1\setminus S_2} G_{,k}(x,\xi_0)n_k(x)\,\mathrm{d}S(x). \end{aligned}$$

The last term above is equal to

$$-\frac{\delta_{rp}}{4\pi}\Omega_{S_1\setminus S_2}(\xi_0),$$

where Ω is the solid angle subtended by the surface $S_1\backslash S_2$ at ξ_0.

In order to use Stokes' theorem [eqn (B1)] one can add and subtract the expression

$$\int_{S_1\setminus S_2} [G_{,k}(x,\xi_0)(x_p-\xi_{0p})]_{,k}n_r(x)\,\mathrm{d}S(x)$$

to the right-hand side of the above equation. Applying Stokes' theorem and using the fact that $G_{,kk}(x,\xi_0) = 0$, one gets

$$B_{rp}(S_1)-B_{rp}(S_2) = -\epsilon_{qkr}\oint_{C_1\setminus C_2} G_{,k}(x,\xi_0)(x_p-\xi_{0p})\,\mathrm{d}x_q-\delta_{kp}\int_{S_1\setminus S_2} G_{,k}(x,\xi_0)n_r(x)\,\mathrm{d}S(x)-\frac{\delta_{rp}}{4\pi}\Omega_{S_1\setminus S_2}(\xi_0).$$

Now, the expression

$$\delta_{kp}\int_{S_1\setminus S_2} G_{,k}(x,\xi_0)n_r(x)\,\mathrm{d}S(x)$$

is added and subtracted to the second integral on the right-hand side of the above equation and Stokes' theorem is applied once more. Finally, one obtains

$$\begin{aligned} B_{rp}(S_1)-B_{rp}(S_2) = -\epsilon_{qkr}\oint_{C_1\setminus C_2} G_{,k}(x,\xi_0)(x_p-\xi_{0p})\,\mathrm{d}x_q-\epsilon_{qrp}\oint_{C_1\setminus C_2} G(x,\xi_0)\,\mathrm{d}x_q \\ -\int_{S_1\setminus S_2} G_{,r}(x,\xi_0)n_p(x)\,\mathrm{d}S(x)-\frac{\delta_{rp}}{4\pi}\Omega_{S_1\setminus S_2}(\xi_0). \end{aligned} \tag{B4}$$

Determination of $C_r(S_1)$

$$\begin{aligned} C_r(S_1)-C_r(S_2) &= \int_{S_1\setminus S_2} \frac{\partial G(x,\xi_0)}{\partial \xi_r}\,\mathrm{d}S(x) \\ &= -\int_{S_1\setminus S_2} G_{,r}(x,\xi_0)\,\mathrm{d}S(x). \end{aligned} \tag{B5}$$

Final result

Using eqns (B4) and (B5),

$$\begin{aligned} n_p(\xi_0)C_r(S_1)-B_{rp}(S_1) = \epsilon_{qkr}\oint_{C_1} G_{,k}(x,\xi_0)(x_p-\xi_{0p})\,\mathrm{d}x_q+\epsilon_{qrp}\oint_{C_1} G(x,\xi_0)\,\mathrm{d}x_q \\ +\int_{S_1} G_{,r}(x,\xi_0)[n_p(x)-n_p(\xi_0)]\,\mathrm{d}S(x)+\frac{\delta_{rp}}{4\pi}\Omega_{S_1}(\xi_0). \end{aligned} \tag{B6}$$

The above equation contains, at most, weakly singular integrals, provided that $n(x)$ is of class $C^{0,\beta}$ at ξ_0 for some $\beta > 0$.

Use of eqns (B3) and (B6) in eqn (B2) results in eqn (28).